

# Introduction to co-split Lie algebras

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## Abstract

In this work, we introduce a new concept which is obtained by defining a new compatibility condition between Lie algebras and Lie coalgebras. With this terminology, we describe the interrelation between the Killing form and the adjoint representation in a new perspective.

## 1 Introduction

During the past decade, there have appeared a number of papers on the study of Lie bialgebras (see [EK], [ES] and references therein, etc). It is well-known that a Lie bialgebra is a vector space endowed simultaneously with a Lie algebra structure and a Lie coalgebra structure, together with a certain compatibility condition, which was suggested by a study of Hamiltonian mechanics and Poisson Lie groups ([ES]).

In the present work, we consider a new [Lie algebra]-[Lie coalgebra] structure, say, a co-split Liealgebra. Using this concept, we can easily study the Lie algebra structure on the dual space of a semi-simple Lie algebra from another point of view.

This paper is arranged as follows: At first we recall some concepts and study the relations between Lie algebras and Lie coalgebras. Then we give the definition of a co-split Lie algebra. In section 4, we prove that  $sl_{n+1}(\mathbf{C})$  is a co-split Lie algebra. Then we discuss the interrelation of the Killing form and the adjoint representation of  $sl_{n+1}(\mathbf{C})$ . Finally, the results are proved to hold for all finite dimensional complex semi-simple Lie algebras.

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## 2 Basics

In this section, we mainly recall the definitions of Lie algebras, Lie coalgebras and Lie bialgebras, and also their relationship. For more information, one can see [EK], [ES] and references therein.

A Lie algebra is a pair  $(L, [\cdot, \cdot])$ , where  $L$  is a linear space and  $[\cdot, \cdot] : L \times L \longrightarrow L$  is a bilinear map (in fact, it is a linear map from  $L \otimes L$  to  $L$ ) satisfying:

- (L1)  $[a, b] + [b, a] = 0$ ,
- (L2)  $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$ .

For any spaces  $U, V, W$ , define maps

$$\begin{aligned} \tau : \quad U \otimes V &\longrightarrow V \otimes U \\ u \otimes v &\longmapsto v \otimes u, \\ \xi : \quad U \otimes V \otimes W &\longrightarrow V \otimes W \otimes U \\ u \otimes v \otimes w &\longmapsto v \otimes w \otimes u. \end{aligned}$$

A Lie coalgebra is a pair  $(L, \delta)$ , where  $L$  is a linear space and  $\delta : L \longrightarrow L \otimes L$  is a linear map satisfying:

- (Lc1)  $(1 + \tau) \circ \delta = 0$ ,
- (Lc2)  $(1 + \xi + \xi^2) \circ (1 \otimes \delta) \circ \delta = 0$ .

A Lie bialgebra is a triple  $(L, [\cdot, \cdot], \delta)$  such that

- (Lb1)  $(L, [\cdot, \cdot])$  is a Lie algebra,
- (Lb2)  $(L, \delta)$  is a Lie coalgebra,
- (Lb3) For any  $x, y \in L$ ,  $\delta([x, y]) = x \cdot \delta(y) - y \cdot \delta(x)$ .

The compatibility condition (Lb3) shows that  $\delta$  is a derivation map.

In the following lemmas,  $c$  is an arbitrary constant.

**Lemma 2.1** *For any finite dimensional Lie algebra  $(L, [\cdot, \cdot])$ , the dual space  $L^*$  has a Lie coalgebra structure defined by*

$$\delta_{L^*}(f^*) = \sum_{(f)} f_1 \otimes f_2 : x \otimes y \longmapsto f_1(x)f_2(y) = cf^*([x, y]).$$

**Lemma 2.2** *For any finite dimensional Lie coalgebra  $(L, \delta)$ , the dual space  $L^*$  has a Lie algebra structure defined by*

$$[f^*, g^*] : x \longmapsto cf^*(x_1)g^*(x_2),$$

where  $\delta(x) = \sum_{(x)} x_1 \otimes x_2$ .

These two lemmas are natural conclusions and easy to be verified.

## 3 What is a co-split Lie

**Definition 3.1** *Suppose that  $(L, [\cdot, \cdot])$  is a Lie algebra and  $(L, \delta)$  is a Lie coalgebra. A triple  $(L, [\cdot, \cdot], \delta)$  is called a co-split Lie if the compatibility condition*

$$[\cdot, \cdot] \circ \delta = \text{id}_L$$

holds, and  $\delta$  is called a co-splitting of  $L$ .

If in the compatibility condition,  $\text{id}_L$  is replaced by a non-degenerate diagonal matrix, then  $(L, [, ], \delta)$  is called a weak co-split Liealgebra and  $\delta$  is called a weak co-splitting .

**Remark 3.1** Obviously, a co-split LieL should satisfies  $[L, L] = L$ .

**Remark 3.2** If  $(L, [, ], \delta)$  is a finite dimensional (weak) co-split Liealgebra, so is  $(L^*, \delta^*, [, ]^*)$ , where

$$\begin{aligned}\delta^*(f \otimes g)(x) &= (f \otimes g)\delta(x), \\ [, ]^*(f)(x \otimes y) &= f([x, y]),\end{aligned}$$

for all  $x, y \in L$  and  $f, g \in L^*$ . This follows from the fact that  $V \longrightarrow V^*$  is a contravariant functor.

## 4 Co-split Lie algebras of type A

Suppose that  $L$  is a complex simple Lie algebra of type  $A_n$ , then it can be realized as the special linear Lie algebra  $sl_{n+1}(\mathbf{C})$  with basis

$$\{E_{i,j}, E_{j,i}, E_{i,i} - E_{j,j} \mid 1 \leq i < j \leq n+1\}.$$

The Lie bracket is the commutator

$$[E_{i,j}, E_{k,l}] = \delta_{j,k}E_{i,l} - \delta_{l,i}E_{k,j}.$$

Define a linear map  $\delta : sl_{n+1}(\mathbf{C}) \longrightarrow sl_{n+1}(\mathbf{C}) \otimes sl_{n+1}(\mathbf{C})$  as

$$\delta(E_{i,j}) = \frac{1}{2n+2} \sum_{k=1}^{n+1} (E_{i,k} \otimes E_{k,j} - E_{k,j} \otimes E_{i,k}).$$

**Proposition 4.1**  $\delta$  is well-defined.

*Proof.* Assume that  $i \neq j$ , then

$$\begin{aligned}\delta(E_{i,j}) &= \frac{1}{2n+2} \sum_{k=1}^{n+1} (E_{i,k} \otimes E_{k,j} - E_{k,j} \otimes E_{i,k}) \\ &= \frac{1}{2n+2} \sum_{k \neq i,j} (E_{i,k} \otimes E_{k,j} - E_{k,j} \otimes E_{i,k}) \\ &\quad + \frac{1}{2n+2} [(E_{i,i} - E_{j,j}) \otimes E_{i,j} - E_{i,j} \otimes (E_{i,i} - E_{j,j})].\end{aligned}$$

$$\begin{aligned}
\delta(E_{i,i} - E_{j,j}) &= \frac{1}{2n+2} \sum_{k=1}^{n+1} (E_{i,k} \otimes E_{k,i} - E_{j,k} \otimes E_{k,j}) \\
&\quad - \frac{1}{2n+2} \sum_{k=1}^{n+1} (E_{k,i} \otimes E_{i,k} - E_{k,j} \otimes E_{j,k}) \\
&= \frac{1}{2n+2} \sum_{k \neq i} (E_{i,k} \otimes E_{k,i} - E_{k,i} \otimes E_{i,k}) \\
&\quad - \frac{1}{2n+2} \sum_{k \neq j} (E_{j,k} \otimes E_{k,j} - E_{k,j} \otimes E_{j,k}).
\end{aligned}$$

Hence  $\delta$  is well-defined.  $\square$

**Theorem 4.1**  $(sl_{n+1}(\mathbf{C}), \delta)$  is a Lie coalgebra.

*Proof.* At first, it is clear that  $(1 + \tau) \circ \delta = 0$ . By a direct calculation, we have

$$\begin{aligned}
&((1 \otimes \delta) \circ \delta)(E_{i,j}) \\
&= \frac{1}{2n+2} (1 \otimes \delta) \left( \sum_{k=1}^{n+1} (E_{i,k} \otimes E_{k,j} - E_{k,j} \otimes E_{i,k}) \right) \\
&= \frac{1}{4(n+1)^2} \sum_{1 \leq k, l \leq n+1} (E_{i,k} \otimes E_{k,l} \otimes E_{l,j} - E_{i,k} \otimes E_{l,j} \otimes E_{k,l}) \\
&\quad - \frac{1}{4(n+1)^2} \sum_{1 \leq k, l \leq n+1} (E_{k,j} \otimes E_{i,l} \otimes E_{l,k} - E_{k,j} \otimes E_{l,k} \otimes E_{i,l}) \\
&= \frac{1}{4(n+1)^2} \sum_{1 \leq k, l \leq n+1} (E_{i,k} \otimes E_{k,l} \otimes E_{l,j} - E_{l,j} \otimes E_{i,k} \otimes E_{k,l}) \\
&\quad - \frac{1}{4(n+1)^2} \sum_{1 \leq k, l \leq n+1} (E_{i,l} \otimes E_{k,j} \otimes E_{l,k} - E_{k,j} \otimes E_{l,k} \otimes E_{i,l}).
\end{aligned}$$

Hence,

$$(1 + \xi + \xi^2) \circ (1 \otimes \delta) \circ \delta = 0,$$

that is,  $\delta$  satisfies the anti-symmetry property and the Jacobi identity. Then  $(sl_{n+1}(\mathbf{C}), \delta)$  is a Lie coalgebra.  $\square$

**Theorem 4.2**  $(sl_{n+1}(\mathbf{C}), [\cdot, \cdot], \delta)$  is a co-split Lie algebra.

*Proof.* For  $i \neq j$ , it is easy to check that

$$\begin{aligned}
([\cdot, \cdot] \circ \delta)(E_{i,j}) &= \frac{1}{2n+2} \sum_{k=1}^{n+1} ([E_{i,k}, E_{k,j}] - [E_{k,j}, E_{i,k}]) \\
&= E_{i,j},
\end{aligned}$$

$$\begin{aligned}
([\cdot, \cdot] \circ \delta)(E_{i,i} - E_{j,j}) &= \frac{1}{2n+2} \sum_{k=1}^{n+1} ([E_{i,k}, E_{k,i}] - [E_{k,i}, E_{i,k}]) \\
&\quad - \frac{1}{2n+2} \sum_{k=1}^{n+1} ([E_{j,k}, E_{k,j}] - [E_{k,j}, E_{j,k}]) \\
&= E_{i,i} - E_{j,j},
\end{aligned}$$

that is,  $[\cdot, \cdot] \circ \delta = \text{id}$ , also by Theorem 4.1. So, the theorem holds.  $\square$

## 5 Dual Lie algebras, Killing form and adjoint representation

In this section, we discuss the interrelation of the Killing form and the adjoint representation for the Lie algebra of type  $A$  within our new terminology.

**Theorem 5.1**  $((sl_n)^*, -2n\delta^*)$  is a Lie algebra isomorphic to  $sl_n$ , the isomorphism is given by

$$B : f_{i,j} \mapsto E_{j,i},$$

where  $\{f_{i,j} \mid 1 \leq i, j \leq n\}$  forms a basis of  $(gl_n)^* \supset (sl_n)^*$ , and

$$f_{i,j}(E_{k,l}) = \delta_{i,k} \delta_{j,l}.$$

*Proof.* By definition, we have

$$\begin{aligned}
-2n\delta^*(f_{i,j} \otimes f_{k,l})(E_{s,t}) &= - \sum_{r=1}^n (f_{i,j}(E_{s,r}) f_{k,l}(E_{r,t}) - f_{i,j}(E_{r,t}) f_{k,l}(E_{s,r})) \\
&= -\delta_{j,k} (f_{i,j}(E_{s,j}) f_{j,l}(E_{j,t}) + \delta_{i,l} f_{i,j}(E_{i,t}) f_{k,i}(E_{s,i})) \\
&= -(\delta_{j,k} f_{i,l} - \delta_{i,l} f_{k,j})(E_{s,t}),
\end{aligned}$$

then  $(sl_n)^*$  is a Lie algebra under bracket  $-2n\delta^*$ , and  $B$  is an isomorphism.  $\square$

Define a bilinear form  $(\cdot, \cdot)_B : sl_n \times sl_n \longrightarrow \mathbf{C}$  as  $(x, y)_B = B^{-1}(x)(y)$ .

**Theorem 5.2**  $(\cdot, \cdot)_B$  is just a non-zero scalar of the Killing form.

*Proof.* This result is direct.  $\square$

Now we can consider the following maps:

$$sl_n \xrightarrow{2n\delta} sl_n \otimes sl_n \xrightarrow{\text{id}_{sl_n} \otimes B^{-1}} sl_n \otimes (sl_n)^* \xrightarrow{\eta} \mathbf{End}(sl_n)$$

where  $\eta$  is an isomorphism,  $\eta(x \otimes f)(y) = f(y)x$ .

**Theorem 5.3** For the adjoint representation

$$\text{ad} : sl_n \longrightarrow \mathbf{End}(sl_n),$$

we have

$$\text{ad} = 2n\eta \circ (\text{id}_{sl_n} \otimes B^{-1}) \circ \delta.$$

*Proof.* For any  $E_{i,j}, E_{k,l}$ , we have

$$\begin{aligned}
2n\eta \circ (\text{id}_{sl_n} \otimes B^{-1}) \circ \delta(E_{i,j})(E_{k,l}) &= \sum_{s=1}^n (f_{j,s}(E_{k,l})E_{i,s} - f_{s,i}(E_{k,l})E_{s,j}) \\
&= \delta_{j,k}E_{i,l} - \delta_{i,l}E_{k,j} \\
&= \text{ad}(E_{i,j})(E_{k,l}).
\end{aligned}$$

□

**Remark 5.1** For convenience, many computations are made in  $gl_n$  or  $(gl_n)^*$ , but the results always hold in  $sl_n$  or  $(sl_n)^*$ .

## 6 Co-splitting Theorem

In this section, we prove the following theorem:

**Theorem 6.1** *Any finite dimensional complex simple Lie algebra has a co-split Lie structure.*

*Proof.* For a simple Lie algebra  $L$  of type  $X_l$  rather than of type  $A$ , our proof is divided into following steps.

### Step 1:

Suppose that  $V$  is a non-trivial irreducible  $X_l$ -module of dimension  $n$ . Then there is an injection

$$\rho : L \longrightarrow sl_n \subset \mathbf{End}(V),$$

and it is easy to check that the bilinear form  $(\cdot, \cdot)_B$  of  $sl_n$  is still non-degenerate over  $\rho(L)$ .

### Step 2:

Let  $M$  be the orthogonal complement of  $\rho(L)$  with respect to  $(\cdot, \cdot)_B$ , that is,

$$M = \{m \in sl_n \mid (m, \rho(L))_B = 0\}.$$

Then  $M$  is a  $\rho(L)$ -submodule and  $sl_n = \rho(L) \oplus M$ .

### Step 3:

For any element  $a = x + v \in sl_n \otimes sl_n$ , if we have  $x \in \rho(L) \otimes \rho(L)$  and  $v \in \rho(L) \otimes M + M \otimes \rho(L) + M \otimes M$ , the projective map  $\mathbf{Proj}_{\rho(L) \otimes \rho(L)}^{sl_n \otimes sl_n}$  is defined to map  $a$  to  $x$ . Now we write  $\delta_{res} =: \mathbf{Proj}_{\rho(L) \otimes \rho(L)}^{sl_n \otimes sl_n} \circ \delta|_{\rho(L)}$ , where  $\delta$  is given in Section 4., then we have

**Lemma 6.1**  $(\rho(L), \delta_{res})$  is a Lie coalgebra.

*Proof.* At first, it is easy to show that  $\delta$  is an injective map of  $sl_n$ -module, hence of  $\rho(L)$ -modules.

By Theorem 5.3,  $\delta$  is equivalent to the adjoint representation, so it is easy to know that  $\delta(\rho(L)) \subset \rho(L) \otimes \rho(L) + M \otimes M \cong \rho(L) \otimes \rho(L)^* + M \otimes M^*$ . Now the skew-symmetry of  $\delta_{res}$  is clear.

Furthermore, we have

$$(1 \otimes \delta_{res}) \circ \delta_{res} = \mathbf{Proj}_{\rho(L) \otimes \rho(L) \otimes \rho(L)}^{sl_n \otimes sl_n \otimes sl_n} \circ [(1 \otimes \delta) \circ \delta]|_{\rho(L)},$$

it is obvious by the contained relation

$$(1 \otimes \delta) \circ \delta(\rho(L)) \subset \rho(L) \otimes \rho(L) \otimes \rho(L) + \rho(L) \otimes M \otimes M + M \otimes \rho(L) \otimes M + M \otimes M \otimes \rho(L),$$

thus we have proved this lemma.  $\square$

#### Step 4:

##### Lemma 6.2

$$[, ] \circ \delta_{res} = a \text{ non-zero scalar of } \text{id}_{\rho(L)}.$$

*Proof.* Suppose that  $\Delta^+$  is the positive root system of  $X_l$  and  $\gamma$  is the highest root. It is easy to find a basis of  $\rho(L)$

$$\{X_{\pm\alpha}, h_i \mid i = 1, \dots, l; \alpha \in \Delta^+\}$$

such that  $(h_i, h_j)_B = \delta_{i,j}$  and  $\alpha(h_i) \in \mathbf{R}$ .

Since  $\gamma$  is the highest root, then for any  $\alpha \in \Delta^+$ ,  $[E_\gamma, E_\alpha] = 0$ . By the property of  $\delta$  (Theorem 5.3) and definition of  $\delta_{res}$ , we have

$$2n\delta_{res}(X_\gamma) = \sum_{i=1}^l [X_\gamma, h_i] \otimes h_i + \sum_{\alpha \in \Delta^+} [X_\gamma, X_{-\alpha}] \otimes \frac{X_\alpha}{(X_\alpha, X_{-\alpha})_B},$$

hence

$$\begin{aligned} 2n[, ] \circ \delta_{res}(X_\gamma) &= \sum_{i=1}^l [[X_\gamma, h_i], h_i] + \sum_{\alpha \in \Delta^+} \frac{[[X_\gamma, X_{-\alpha}], X_\alpha]}{(X_\alpha, X_{-\alpha})_B} \\ &= \left[ \sum_{i=1}^l \gamma(h_i)^2 + \sum_{\alpha \in \Delta^+} \gamma(h_\alpha) \right] X_\gamma, \end{aligned}$$

where

$$h_\alpha = [X_\alpha, X_{-\alpha}]/(X_\alpha, X_{-\alpha})_B = \frac{2\alpha}{(\alpha, \alpha)},$$

the second assertion holds by  $\rho(L) \cong \rho(L)^*$ .

Clearly,

$$\sum_{i=1}^l \gamma(h_i)^2 + \sum_{\alpha \in \Delta^+} \gamma(h_\alpha) > 0,$$

then  $[, ] \circ \delta_{res}(X_\gamma) \neq 0$ .

Secondly,  $\delta_{res}(X_\gamma)$  is a highest weight vector of  $L$ -module  $\rho(L) \otimes \rho(L) \cong L \otimes L$ , thus the equation in this lemma holds.  $\square$

Up to now, we have completed the proof of Theorem 6.1.

We also obtain the following result.

**Theorem 6.2** *For any type  $X_L$ , we have*

$$2n\eta \circ (\text{id}_{sl_n} \otimes B^{-1}) \circ (\delta_{res}) = \text{ad}_{\rho(L)|_{\rho(L)}},$$

where  $\text{ad}_{\rho(L)|_{\rho(L)}}(\rho(x) + m) = \text{ad}_{(\rho(x))|_{\rho(L)}}$  for any  $x \in L, m \in M$ .

*Proof.* For any  $x, y \in L$ , by the obvious fact  $[\rho(x), \rho(y)] \in \rho(L)$ , we have

$$\begin{aligned} & [2n\eta \circ (\text{id}_{sl_n} \otimes B^{-1}) \circ \delta](\rho(x))(\rho(y)) \\ &= [2n\eta \circ (\text{id}_{sl_n} \otimes B^{-1}) \circ (\delta_{res})](\rho(x))(\rho(y)) \\ &= \text{ad}_{(\rho(x))}(\rho(y)) \\ &= \text{ad}_{\rho(L)(\rho(x))|_{\rho(L)}}(\rho(y)), \end{aligned}$$

and by the definition of  $\delta_{res}$ ,

$$[2n\eta \circ (\text{id}_{sl_n} \otimes B^{-1}) \circ (\delta_{res})](\rho(x))(M) = 0.$$

So, the claim is true.  $\square$

**Remark 6.1** *This work shows that for any finite dimensional semi-simple Lie algebra  $L$  over the complex field  $\mathbf{C}$  (or, equivalently, over any algebraically closed field with characteristic zero), there exists some important relation between its Killing form and adjoint action. Hence our new algebraic structure is proved to be very useful. However, much more problems about it need to be solved.*

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